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Verblunsky coefficients with Coulomb-type decay

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Dedicated to Barry Simon on the occasion of his 60th birthday

Abstract

We show that probability measures on the unit circle associated with Verblunsky coefficients obeying a Coulomb-type decay estimate have no singular continuous component.

1. Introduction

Let $d\mu$ be a probability measure on $\mathbb{R}/(2\pi\mathbb{Z})$ that is not supported on a finite number of points. Then, using the Gram–Schmidt procedure, we may find polynomials $\varphi_n(z)$ that obey

$$\int_0^{2\pi} \overline{\varphi_m(e^{i\eta})}\varphi_n(e^{i\eta})\, d\mu(\eta) = \delta_{m,n}.$$

We also consider the monic orthogonal polynomials $\Phi_n(z)$. They obey the Szegő recursion

$$\Phi_{n+1}(z) = z\Phi_n(z) - \overline{\alpha}_n \Phi_n^*(z),$$

where $\Phi_n^*(z) = z^n\overline{\Phi_n(1/\overline{z})}$. The α_n are called Verblunsky coefficients and they belong to the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Conversely, every $\alpha \in \times_{n=0}^{\infty}\mathbb{D}$ corresponds to a unique measure. See [16–18] for background material on orthogonal polynomials on the unit circle (OPUC).

In this paper we are interested in the measures associated with Verblunsky coefficients that have Coulomb-type decay. To motivate our study, let us recall the following result of Golinskii and Ibragimov [4]:

$$\sum_{n=0}^{\infty}(n+1)|\alpha_n|^2 < \infty \quad \Rightarrow \quad d\mu_{\text{sing}} = 0.$$

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Here, $d\mu_{\rm sing} = d\mu_{\rm sc} + d\mu_{\rm pp}$, where $d\mu = d\mu_{\rm ac} + d\mu_{\rm sc} + d\mu_{\rm pp}$ is the Lebesgue decomposition of $d\mu$ into an absolutely continuous (with respect to Lebesgue measure) piece, a singular continuous piece, and a pure point piece.

The natural class of Verblunsky coefficients having true Coulomb decay, that is, $|\alpha_n| = O(1/n)$, is outside the scope of the result above. More generally, one may be interested in the class of Verblunsky coefficients satisfying

$$\sum_{n=0}^{N} (n+1)|\alpha_n|^2 \leqslant A \log N \tag{1}$$

for some $A < \infty$. The following extension, due to Simon [15], of the result of Golinskii and Ibragimov covers a portion of this class:

$$\alpha \text{ satisfies (1) for some } A < \tfrac{1}{4} \quad \Rightarrow \quad d\mu_{\rm sing} = 0. \tag{2}$$

Simon also shows that for every $A > \frac{1}{4}$, there is an example satisfying (1) with $d\mu_{\rm pp} \neq 0$. Thus, the result (2) is almost sharp. The latter result is an OPUC analogue of the classical Wigner–von Neumann example that exhibits an embedded eigenvalue for a half-line Schrödinger operator with $O(1/x)$ potential [19].

The pure point component is further studied in [17]. There it is shown (see Theorem 10.12.7) that if (1) holds for some A, then $d\mu$ has at most K pure points, where K is the unique integer with $K \leqslant 4A < K + 1$. Following this theorem, Simon writes that it is an intriguing open question if (1) implies $d\mu_{\rm sc} = 0$. There are two reasons why one expects a positive answer to this question. Intuitively, it should be easier to have infinitely many pure points than a singular continuous component, so that the result just quoted supports the conjecture that a singular continuous piece should be impossible. On the other hand, Kiselev has proven the absence of singular continuous spectrum for half-line Schrödinger operators with $O(1/x)$ potentials [5].

Our goal here is to give an affirmative answer to Simon's question and prove the following theorem:

Theorem 1. *Suppose there is $A < \infty$ such that α satisfies* (1). *Then*, $d\mu_{\rm sc} = 0$.

Since it is also shown in [16] (see Corollary 2.7.6) that (1) with $A = 1/4$ implies $d\mu_{\rm pp} = 0$, it follows from Theorem 1 that (2) may be strengthened to

$$\alpha \text{ satisfies (1) for some } A \leqslant \tfrac{1}{4} \quad \Rightarrow \quad d\mu_{\rm sing} = 0,$$

which is optimal by the discussion above.

The overall strategy in our proof of Theorem 1 will be inspired by Kiselev [5]. This will require some preparatory work. We first recall Prüfer variables and the Bernstein–Szegő Approximation to $d\mu$ in Section 2 and prove a comparison lemma which is related to the Chebyshev–Markov Moment Problem. Then, we consider the support of $d\mu_{\rm sing}$ in Section 3 and prove that it has Hausdorff dimension zero. This is a result in the spirit of Remling [12] who proved results of this flavor for half-line Schrödinger operators. Finally, we prove Theorem 1 in Section 4 by working out the OPUC analogue of Kiselev's ideas from [5].

2. Prüfer variables and Bernstein–Szegő approximation

Let $\{\alpha_n\}$ be the Verblunsky coefficients of a non-trivial probability measure $d\mu$ on $\partial\mathbb{D}$. As mentioned above, the α's give rise to a sequence $\{\Phi_n(z)\}$ of monic polynomials (via the Szegő recursion) that are orthogonal with respect to $d\mu$. For $\beta \in [0, 2\pi)$, we also consider the monic polynomials $\{\Phi_n(z, \beta)\}$ that are associated in the same way with the Verblunsky coefficients $\{e^{i\beta}\alpha_n\}$.

Let $\eta \in [0, 2\pi)$. Define the Prüfer variables by

$$\Phi_n(e^{i\eta}, \beta) = R_n(\eta, \beta) \exp\left[i(n\eta + \theta_n(\eta, \beta))\right],$$

where $R_n > 0$, $\theta_n \in [0, 2\pi)$, and $|\theta_{n+1} - \theta_n| < \pi$; compare [9,10,17]. These variables obey the following pair of equations:

$$\frac{R_{n+1}^2(\eta, \beta)}{R_n^2(\eta, \beta)} = 1 + |\alpha_n|^2 - 2\mathrm{Re}\left(\alpha_n e^{i[(n+1)\eta+\beta+2\theta_n(\eta,\beta)]}\right),$$

$$e^{-i(\theta_{n+1}(\eta,\beta)-\theta_n(\eta,\beta))} = \frac{1 - \alpha_n e^{i[(n+1)\eta+\beta+2\theta_n(\eta,\beta)]}}{\left[1 + |\alpha_n|^2 - 2\mathrm{Re}\left(\alpha_n e^{i[(n+1)\eta+\beta+2\theta_n(\eta,\beta)]}\right)\right]^{1/2}}.$$

We also define $r_n(\eta, \beta) = |\varphi_n(\eta, \beta)|$.

When $\{\alpha_n\} \in \ell^2$,

$$r_n(\eta, \beta) \sim R_n(\eta, \beta) \sim \exp\left(-\sum_{j=0}^{n-1} \mathrm{Re}(\alpha_j e^{i[(j+1)\eta+\beta+2\theta_j(\eta,\beta)]})\right). \tag{3}$$

(We write $f_n \sim g_n$ if there is $C > 1$ such that $C^{-1} g_n \leqslant f_n \leqslant C g_n$ for all n.) For the Prüfer equations and (3), see [17, Theorems 10.12.1 and 10.12.3].

Next we recall the Bernstein–Szegő Approximation of $d\mu$. The measure $d\mu_n$ associated with Verblunksy coefficients $\alpha_0, \ldots, \alpha_{n-2}, \alpha_{n-1}, 0, 0, \ldots$ is given by

$$d\mu_n(\eta) = \frac{d\eta}{2\pi r_n^2(\eta, 0)}, \tag{4}$$

compare [16, Theorem 1.7.8].

If $d\mu$ and dv are two measures whose first n Verblunsky coefficients coincide (i.e., $\alpha_k(d\mu) = \alpha_k(dv)$, $0 \leqslant k \leqslant n-1$), their moments up to order n are the same (see, e.g. [16, Theorem 1.5.5.(ii)]). Consequently, given a Laurent polynomial, $f(\eta) = \sum_{k=-n}^{n} f_k e^{ik\eta}$, we have

$$\int_0^{2\pi} f(e^{i\eta})\, d\mu(\eta) = \int_0^{2\pi} f(e^{i\eta})\, dv(\eta). \tag{5}$$

Lemma 2.1. *Suppose $d\mu$ and dv are two measures whose first n Verblunsky coefficients coincide. For every $\kappa > 0$ and every interval $I \subseteq \partial\mathbb{D}$ of length $\delta \geqslant n^{-1/(2+\kappa)}$, we have*

$$\mu(I) \leqslant v(3I) + C\delta^\kappa. \tag{6}$$

Remarks. (a) In (6), $3I$ denotes the interval of length 3δ that has the same center as I and C is a constant that depends only on κ. Alternatively, one may choose a universal C for which (6) holds for all $\delta \leqslant \delta_0$ (and hence $n \geqslant n_0(\kappa)$).

(b) Since the estimate (6) is sufficient for our purpose and has a short and elementary proof, we content ourselves with this explicit statement. We do want to point out, however, that it is closely related to the Chebyshev–Markov Moment Problem: if we fix n initial moments and an interval I, what are the extremal values of $\mu(I)$ when μ ranges over all measures that have the prescribed moments? A wealth of material dealing with this problem may be found, for example, in [7,8]. An analogue of these classical results for Schrödinger operators in $L^2(0,\infty)$ was recently found in [13].

Proof. Without loss of generality, we assume that $I = (-\frac{\delta}{2}, \frac{\delta}{2})$. Consider the Fejér kernel,

$$F_n(\eta) = \sum_{k=-n}^{n} \left(1 - \frac{|k|}{n+1}\right) e^{ik\eta} = \frac{1}{n+1} \left(\frac{\sin \frac{n+1}{2}\eta}{\sin \frac{1}{2}\eta} \right)^2,$$

and let

$$\sigma_n(\eta) = (F_n * \chi_{2I})(\eta) = \frac{1}{2\pi} \int_0^{2\pi} F_n(\tau) \chi_{2I}(\eta - \tau)\, d\tau,$$

where χ_{2I} is the characteristic function of the interval $(-\delta, \delta)$.

Clearly,

$$|\sigma_n(\eta)| \leqslant 1 \quad \text{for all } \eta. \tag{7}$$

Moreover, by (5), it follows that

$$\int_0^{2\pi} \sigma_n(\eta)\, d\mu(\eta) = \int_0^{2\pi} \sigma_n(\eta)\, d\nu(\eta). \tag{8}$$

Note that

$$\sigma_n(\eta) - \chi_{2I}(\eta) = \frac{1}{\pi} \int_0^{\pi} F_n(\tau) \left[\frac{\chi_{2I}(\eta-\tau) + \chi_{2I}(\eta+\tau)}{2} - \chi_{2I}(\eta) \right] d\tau.$$

When $||\eta| - \delta| \geqslant \frac{\delta}{2}$, this gives

$$\sigma_n(\eta) - \chi_{2I}(\eta) = \frac{1}{\pi} \int_{\delta/2}^{\pi} F_n(\tau) \left[\frac{\chi_{2I}(\eta-\tau) + \chi_{2I}(\eta+\tau)}{2} - \chi_{2I}(\eta) \right] d\tau.$$

Consequently, for these values of η, we have

$$\left|\sigma_n(\eta) - \chi_{2I}(\eta)\right| \leqslant \frac{2}{\pi} \int_{\delta/2}^{\pi} F_n(\tau)\, d\tau \leqslant \frac{2}{n+1} \frac{1}{\sin^2 \frac{\delta}{2}} \lesssim \frac{1}{\delta^2 n} \leqslant \delta^{\kappa},$$

where we used the assumption $\delta \geqslant n^{-1/(2+\kappa)}$ in the last step. [1] Thus,

$$|\sigma_n(\eta) - \chi_{2I}(\eta)| \lesssim \delta^{\kappa} \quad \text{for all } \eta \text{ satisfying } ||\eta| - \delta| \geqslant \frac{\delta}{2}. \tag{9}$$

The assertion of the lemma is an immediate consequence of (7)–(9). □

[1] We write $f \lesssim g$ if $f \leqslant Cg$ for a suitable constant C.

3. Zero-dimensionality of the singular part

In this section we show that the singular part of $d\mu$ must be supported on a set of zero Hausdorff dimension if the Verblunsky coefficients obey (1). Results of this kind were obtained in the context of Schrödinger operators by Remling [12], Christ and Kiselev [2], and Damanik and Killip [3], for example. We will follow ideas from [3] rather closely.

Theorem 2. *If the Verblunsky coefficients* $\{\alpha_n\}$ *satisfy* (1), *then the set*

$$S = \{\eta \in [0, 2\pi) : R_n(\eta, \beta) \text{ is unbounded for some } \beta\}$$

has zero Hausdorff dimension.

The following consequence of Theorem 2 is central to our proof of Theorem 1:

Corollary 3.1. *If the Verblunsky coefficients* $\{\alpha_n\}$ *satisfy* (1), *then* $d\mu_{\rm sing}$ *is supported on a set of zero Hausdorff dimension.*

Proof. It was shown in [17, Corollary 10.8.4] that $d\mu_{\rm sing}$ is supported on the set S. □

Assuming (1), we see that

$$\sum_{n=1}^{\infty} |\alpha_n|^2 \leqslant \sum_{k=0}^{\infty} 2^{-k} \sum_{n=2^k}^{2^{k+1}-1} (n+1)|\alpha_n|^2 \leqslant A \log 2 \sum_{k=0}^{\infty} (k+1) 2^{-k}.$$

Thus, (1) implies $\{\alpha_n\} \in \ell^2$ and, because of (3), our goal is hence to show that

$$A(n, \eta, \beta) = \sum_{j=0}^{n-1} \alpha_j e^{i[(j+1)\eta + \beta + 2\theta_j(\eta, \beta)]}$$

is a bounded function of n for all β, provided that η is away from a set of zero Hausdorff dimension.

Lemma 3.2. *If*

$$\hat{\alpha}(\eta, n) = \lim_{N \to \infty} \sum_{j=n}^{N} \alpha_j e^{ij\eta}$$

exists and obeys

$$\sum_{j=1}^{\infty} |\hat{\alpha}(\eta, j)\alpha_{j-1}| < \infty, \tag{10}$$

then $\eta \notin S$.

Proof. We will show that $A(n, \eta, \beta)$ is bounded (in n) for every $\beta \in [0, 2\pi)$ when (10) holds. The assertion then follows from (3).

Write $\gamma_j(\eta, \beta) = (j+1)\eta + \beta + 2\theta_j(\eta, \beta)$. We have

$$A(n, \eta, \beta) = \sum_{j=0}^{n-1} \left[\hat{\alpha}(\eta, j) - \hat{\alpha}(\eta, j+1)\right] e^{i\gamma_j(\eta,\beta) - ij\eta}$$
$$= \sum_{j=1}^{n-1} \hat{\alpha}(\eta, j) \left[e^{i\gamma_j(\eta,\beta)} - e^{i(\gamma_{j-1}(\eta,\beta)+\eta)}\right] e^{-ij\eta} + O(1).$$

Since

$$|e^{i\gamma_j(\eta,\beta)} - e^{i(\gamma_{j-1}(\eta,\beta)+\eta)}| \leqslant |\gamma_j(\eta, \beta) - \gamma_{j-1}(\eta, \beta) - \eta|$$
$$= 2|\theta_j(\eta, \beta) - \theta_{j-1}(\eta, \beta)|$$
$$\lesssim |\alpha_{j-1}|,$$

boundedness of $A(n, \eta, \beta)$ follows. □

Lemma 3.3. *Let $d\nu$ be a positive measure on $[0, 2\pi)$. For each $\varepsilon \in (0, 1)$ and every measurable function m from $[0, 2\pi)$ to the set of non-negative integers,*

$$\left\{\int \left|\sum_{n=0}^{m(\eta)} \alpha_n e^{-in\eta}\right| d\nu(\eta)\right\}^2 \lesssim \mathcal{E}_\varepsilon(\nu) \sum_{n=0}^{\infty} (n+1)^{1-\varepsilon} \left|\alpha_n\right|^2,$$

where $\mathcal{E}_\varepsilon$ denotes the ε-energy of $d\nu$: $\mathcal{E}_\varepsilon(\nu) = \int\int (1 + |x-y|^{-\varepsilon})\, d\nu(x)\, d\nu(y)$.

Proof. This follows by slightly adjusting the calculation from [20, §XIII.11, p. 196], see also [1, §V.5]. (To deal with the absolute value, one can introduce a phase factor which comes along for the ride as one follows through the steps of the calculation from [20].) □

Proof of Theorem 2. We will apply the criterion of Lemma 3.2. Let us first note that by [20, Theorem XIII.11.3] and [1, §IV.1], the series defining $\hat{\alpha}$ converges off a set of zero Hausdorff dimension. Therefore, we may exclude from consideration those values of η for which $\hat{\alpha}$ is not defined.

By applying the Cauchy–Schwarz inequality to dyadic blocks, for example, we see that (1) implies $n^{-\varepsilon/4}\alpha_n \in \ell^1$ for all $\varepsilon > 0$. Hence the proposition will follow from Lemma 3.2 once we prove that for all $\varepsilon > 0$, the set of η for which $n^{\varepsilon/4}\hat{\alpha}(\eta, n)$ is unbounded is of Hausdorff dimension no more than ε.

Let $m(\eta)$ be a measurable integer-valued function on $[0, 2\pi)$. Because of (1), Lemma 3.3 implies

$$\int \left|\sum_{n=m_l(\eta)}^{2^{l+1}-1} \alpha_n e^{in\eta}\right| d\nu(\eta) = \int \left|\sum_{n=0}^{\tilde{m}_l(\eta)} \alpha_{2^{l+1}-1-n} e^{-in\eta}\right| d\nu(\eta)$$
$$\lesssim \left\{\sum_{n=2^l}^{2^{l+1}-1} (n+1)^{1-\varepsilon} \left|\alpha_n\right|^2\right\}^{1/2} \sqrt{\mathcal{E}_\varepsilon(\nu)}$$
$$\lesssim \sqrt{l}\, 2^{-\varepsilon l/2} \sqrt{\mathcal{E}_\varepsilon(\nu)},$$

where $m_l(\eta) = \max\{m(\eta), 2^l\}$, $\tilde{m}_l(\eta) = \min\{2^l - 1, 2^{l+1} - 1 - m(\eta)\}$, and sums with lower index greater than their upper index are to be treated as zero. Multiplying both sides by $2^{\varepsilon l/4}$, summing this over l, and applying the triangle inequality on the left gives

$$\int \left| m(\eta)^{\varepsilon/4} \sum_{n=m(\eta)}^{\infty} \alpha_n e^{in\eta} \right| d\nu(\eta) \lesssim \sqrt{\mathcal{E}_\varepsilon(\nu)}.$$

That is, for any measurable integer-valued function $m(\eta)$,

$$\int m(\eta)^{\varepsilon/4} \left| \hat{\alpha}(\eta, m(\eta)) \right| d\nu \lesssim \sqrt{\mathcal{E}_\varepsilon(\nu)}.$$

This implies that the set on which $n^{\varepsilon/4}\hat{\alpha}(\eta, n)$ is unbounded must be of zero ε-capacity (i.e., it does not support a measure of finite ε-energy).

As the Hausdorff dimension of sets of zero ε-capacity is less than or equal to ε (see [1, §IV.1]), this completes the proof of the fact that S has zero Hausdorff dimension. □

4. Absence of a singular continuous component

In this section we employ ideas of Kiselev [5] to show that there is no singular continuous component when (1) holds. The preparatory work from the previous section will be crucial.

The first step is to study the number of resonant points on the unit circle at which the Prüfer radius may be large. Using (1) and an almost-orthogonality lemma from [6], we will show that their number must be bounded by an explicit constant.

We first recall [6, Lemma 4.4]:

Lemma 4.1. *Let $e_1, \dots, e_K$ be unit vectors in a Hilbert space $\mathcal{H}$ with*

$$Q = K \sup_{k \neq l} |\langle e_k, e_l \rangle| < 1.$$

Then, for any $g \in \mathcal{H}$,

$$\sum_{l=1}^{K} |\langle g, e_l \rangle|^2 \leqslant (1 + Q)\|g\|^2.$$

Below, the Hilbert spaces in question will be given by $\mathcal{H}_n = \mathbb{C}^n$ with inner product

$$\langle f, g \rangle_{\mathcal{H}_n} = \sum_{j=0}^{n-1} \overline{f(j)} g(j)(1 + j).$$

Recall Abel's formula (summation by parts), which reads

$$\sum_{j=m}^{n} (\delta^+ a)(j) \cdot b(j) = a(n+1) \cdot b(n) - a(m) \cdot b(m-1) - \sum_{j=m}^{n} a(j) \cdot (\delta^- b)(j). \quad (11)$$

Here, a, b are sequences, $(\delta^+ a)(j) = a(j+1) - a(j)$, and $(\delta^- b)(j) = b(j) - b(j-1)$.

Lemma 4.2. *Assume* (1). *If* g *is a real-valued sequence with* $|(\delta^- g)(j)| \leqslant B|\alpha_{j-2}|$ *for a suitable* $B > 0$, *then there is a constant* $C > 0$ *such that, for* $\xi \in (0, \frac{1}{2})$, *we have*

$$\sup_{n \geqslant 1} \left| \sum_{j=1}^{n} j^{-1} e^{i[j\xi + g(j)]} \right| \leqslant C \log(\xi^{-1}). \tag{12}$$

Remark. The proof shows that $\frac{1}{2}$ can be replaced by any number $q \in (0, 1)$. The constant C will then also depend on q.

Proof. Consider some $\xi \in (0, \frac{1}{2})$. Let

$$a(j) = -\sum_{k=j}^{\infty} k^{-1} e^{ik\xi} \quad \text{and} \quad b(j) = e^{ig(j)}.$$

Applying Abel's formula with $\tilde{a}(k) = \sum_{m=0}^{k-1} e^{im\xi}$ and $\tilde{b}(k) = 1/k$, we see that

$$|a(j)| \lesssim \frac{1}{\xi j}. \tag{13}$$

Let us now turn to (12).

Clearly,

$$\left| \sum_{j=1}^{\lceil \xi^{-1} \rceil} j^{-1} e^{\left(i[j\xi + g(j)]\right)} \right| \lesssim \log\left(\xi^{-1}\right). \tag{14}$$

On the other hand, for $n > \xi^{-1} + 1$,

$$\left| \sum_{j=\lceil \xi^{-1} \rceil + 1}^{n} j^{-1} e^{\left(i[j\xi + g(j)]\right)} \right| = \left| \sum_{j=\lceil \xi^{-1} \rceil + 1}^{n} (\delta^+ a)(j) \cdot b(j) \right|,$$

which, by (11), is equal to

$$\left| a(n+1) \cdot b(n) - a(\lceil \xi^{-1} \rceil + 1) \cdot b(\lceil \pi \xi^{-1} \rceil) - \sum_{j=\lceil \xi^{-1} \rceil + 1}^{n} a(j) \cdot (\delta^- b)(j) \right|.$$

By (1), (13), and the assumption on g, $|(\delta^- g)(j)| \leqslant B|\alpha_{j-2}|$, this expression is bounded by $\log\left(\xi^{-1}\right)$ times a constant only depending on A and B. (Split the sum into dyadic blocks, apply Cauchy–Schwarz, and then (1).) Combining this bound with (14), the lemma follows. □

Write $A(n, \eta)$ for $A(n, \eta, 0)$. We will consider situations where there are $\eta_1, \ldots, \eta_K$ such that

$$|A(n, \eta_l)| \geqslant \frac{\log n}{14} \quad \text{for } l = 1, \ldots, K \tag{15}$$

and

$$n^{-1/(3K^2)} \leqslant \min_{k \neq l} d(\eta_k, \eta_l), \quad \max d(\eta_k, \eta_l) < \frac{1}{2}. \tag{16}$$

Our goal is to bound K from above. This is accomplished by the following lemma.

Lemma 4.3. *Assume* (1). *Then there are constants* n_0 *and* $K_{\max} = K_{\max}(A)$ *such that for* $n \geqslant n_0$, *there can be no more than* $K_{\max}$ *points in* $[0, 2\pi)$ *for which* (15) *and* (16) *hold.*

Proof. Consider $\eta_1, \ldots, \eta_K$ for which (15) and (16) hold. Define the following vectors in $\mathcal{H}_n$:

$$e_l(j) = E_n^{-1/2} e^{i[(j+1)\eta_l + 2\theta_j(\eta_l, 0)]}(1+j)^{-1}, \quad 1 \leqslant l \leqslant K.$$

The normalization constant E_n is chosen so that the vectors have norm one. Obviously, $E_n = \log n + O(1)$.

Now, for $k \neq l$, we have for n large enough,

$$\begin{aligned}|\langle e_k, e_l \rangle| &= \frac{1}{E_n} \left| \sum_{j=0}^{n-1} (1+j)^{-1} e^{(-i[(j+1)\eta_k - 2\theta_j(\eta_k,0)] + i[(j+1)\eta_l + 2\theta_j(\eta_l,0)])} \right| \\ &= \frac{1}{E_n} \left| \sum_{j=1}^{n} j^{-1} e^{(i[j(\eta_l - \eta_k) - 2\theta_{j-1}(\eta_k,0) + 2\theta_{j-1}(\eta_l,0)])} \right| \\ &\leqslant \frac{D}{K^2}.\end{aligned}$$

We applied Lemma 4.2 together with (16) in the last step. The constant D depends only on A.

Thus, when $K > D$, we may apply Lemma 4.1 and obtain for any $n \geqslant n_0$ and $g \in \mathcal{H}_n$,

$$\sum_{l=1}^{K} \left| \langle g, e_l \rangle_{\mathcal{H}_n} \right|^2 \leqslant 2 \|g\|^2_{\mathcal{H}_n}. \tag{17}$$

Let us apply (17) to $g = (\alpha_0, \ldots, \alpha_{n-1})$. Due to (1), the right-hand side can be estimated as follows:

$$2\|g\|^2_{\mathcal{H}_n} = 2 \sum_{j=0}^{n-1} |\alpha_j|^2 (j+1) \leqslant 2A \log n.$$

On the other hand, by (15),

$$\left| \langle g, e_l \rangle_{\mathcal{H}_n} \right| = E_n^{-1/2} |A(n, \eta_l)| \geqslant E_n^{-1/2} \frac{\log n}{14}.$$

Consequently, (17) implies that if $K > D$ and $n \geqslant n_0$,

$$\frac{K (\log n)^2}{196 E_n} \leqslant 2A \log n.$$

This shows that $K \leqslant \tilde{D}$, with $\tilde{D}$ roughly being equal to $392A$. Therefore, we must have $K \leqslant \max\{D, \tilde{D}\}$ whenever (15) and (16) hold for $n \geqslant n_0$. $\square$

Let us turn to the proof of the main theorem. Given the results above, we may from now on follow the arguments of Kiselev in [5] quite closely.

Proof of Theorem 1. Assume that the singular continuous part of $d\mu$ is non-trivial. Fix an interval $I \subset [0, 2\pi)$ of length less than $\frac{1}{2}$ such that $\mu_{sc}(I) = \Delta > 0$. Since $d\mu_{sc}$ is continuous, we can achieve that $\mu_{sc}(J)$ is as small as we want if J is any subinterval of I of sufficiently small length.

In particular, we can find $\varepsilon_0 \in (0, 1)$ that satisfies the following conditions ($K_{\max}$ and n_0 are the constants from Lemma 4.3):

(i) $\lceil \varepsilon_0^{-3} \rceil > n_0$.
(ii) $\mu_{sc}(J) < \frac{\Delta}{32K_{\max}^3}$ for all intervals $J \subseteq I$ with $|J| \leqslant \varepsilon_0^{K_{\max}^{-2}}$.
(iii) $\frac{\varepsilon_0^{1/2}}{1-\varepsilon_0^{1/2}} \leqslant \frac{\Delta}{32K_{\max}^3}$.
(iv) The last inequality holds in (18) below.
(v) It is small enough so that we may obtain (19) below.

We say that an interval $J \subset I$ belongs to *scale m* if $|J| = \varepsilon_m := \varepsilon_0^m$. Two intervals of scale m are called *separated* if the distance between their centers exceeds $3\varepsilon_m^{K_{\max}^{-2}}$. An interval J of scale m is called *singular* if $\mu_{sc}(J) > \varepsilon_m^{1/2}$.

We first show that there are no more than $K_{\max}$ separated singular intervals at each scale. Assume that there are $K > K_{\max}$ separated singular intervals of scale m: $J_1, \ldots, J_K$. Let $n_m = \lceil \varepsilon_m^{-3} \rceil$. Recall that $d\mu_{n_m}$ denotes the Bernstein–Szegö approximation of $d\mu$ at level n_m. Using Lemma 2.1, we see that

$$\mu_{n_m}(3J_l) \geqslant \mu(J_l) - C\varepsilon_m > \varepsilon_m^{1/2} - C\varepsilon_m \geqslant \frac{1}{2}\,\varepsilon_m^{1/2}. \tag{18}$$

By (1) and (4),

$$\frac{d\mu_{n_m}}{d\eta}(\eta) \sim R_{n_m}^{-2}(\eta, 0).$$

Thus, there are $\eta_l \in 3J_l$, $1 \leqslant l \leqslant K$, such that $R_{n_m}^{-2}(\eta, 0) \gtrsim \varepsilon_m^{-1/2}$, with a uniform implicit constant. In other words,

$$\text{the estimate (15) holds if } \varepsilon_0 \text{ is small enough.} \tag{19}$$

Moreover, $\min_{k \neq l} d(\eta_k, \eta_l) \geqslant \varepsilon_m^{K_{\max}^{-2}}$ because the intervals $J_1, \ldots, J_K$ are separated. Finally, $\max d(\eta_k, \eta_l) < \frac{1}{2}$ since $|I| < \frac{1}{2}$. Thus, this yields a contradiction to Lemma 4.3.

Now write S_m for the union of all singular intervals at scale m. This set can be covered by at most $8K_{\max}$ intervals of size $\varepsilon_m^{K_{\max}^{-2}}$, or else we can find more than $K_{\max}$ separated singular intervals at scale m. By property (ii) of ε_0, we get

$$\mu_{sc}(S_m) \leqslant 8K_{\max} \times \frac{\Delta}{32K_{\max}^3} = \frac{\Delta}{4K_{\max}^2}$$

for every $m \geqslant 1$. Now consider $m \geqslant K_{\max}^2$ and let $\tilde{m} = \lfloor m K_{\max}^{-2} \rfloor \geqslant 1$. If $J_l^{(m)}$ is a singular interval at scale m that obeys $\mu_{sc}(J_l^{(m)}) > \varepsilon_{\tilde{m}}^{1/2}$, it must a subset of $S_{\tilde{m}}$ since it can clearly be extended to a singular interval at scale $\tilde{m}$. Thus, the set

$$S_m \Big\backslash \bigcup_{l<m} S_l$$

can be covered by at most $8K_{\max}$ intervals of length $\varepsilon_m^{K_{\max}^{-2}}$ and each of these intervals obeys $\mu_{sc}(\cdot) \leqslant \varepsilon_{\tilde{m}}^{1/2}$. Consequently,

$$\mu_{sc}\left(S_m \Big\backslash \bigcup_{l<m} S_l \right) \leqslant 8K_{\max}\varepsilon_{\tilde{m}}^{1/2} .$$

Each $\tilde{m}$ corresponds to $K_{\max}^2$ values of m. Thus,

$$\begin{aligned}
\mu_{sc}\left(\bigcup_{m=1}^{\infty} S_m \right) &\leqslant K_{\max}^2 \times \frac{\Delta}{4K_{\max}^2} + K_{\max}^2 \times \sum_{\tilde{m}=1}^{\infty} 8K_{\max}\varepsilon_{\tilde{m}}^{1/2} \\
&= \frac{\Delta}{4} + \sum_{\tilde{m}=1}^{\infty} 8K_{\max}^3 \varepsilon_0^{\tilde{m}/2} \\
&= \frac{\Delta}{4} + 8K_{\max}^3 \frac{\varepsilon_0^{1/2}}{1-\varepsilon_0^{1/2}} \\
&\leqslant \frac{\Delta}{2}.
\end{aligned}$$

In the last step, we used property (iii) of ε_0.

By zero-dimensionality (cf. Proposition 2), $\mu_{sc}|_I$ is supported by the set

$$D = \left\{ \eta \in I : \limsup_{\delta \to 0} \frac{\mu(k-\delta, k+\delta)}{(2\delta)^{1/2}} = \infty \right\}.$$

See, for example, [14, Theorem 67]. Thus, for each $k \in D$, there is a sequence $\delta_n \to 0$ such that

$$\frac{\mu(k-\delta_n, k+\delta_n)}{(2\delta_n)^{1/2}} \to \infty.$$

For n large, define m_n by

$$\frac{\varepsilon_{m_n}}{2} \geqslant \delta_n > \frac{\varepsilon_{m_n+1}}{2} = \frac{\varepsilon_0 \varepsilon_{m_n}}{2}.$$

We obtain

$$\frac{\mu(k-\varepsilon_{m_n}/2, k+\varepsilon_{m_n}/2)}{(\varepsilon_0\varepsilon_{m_n})^{1/2}} \geqslant \frac{\mu(k-\delta_n, k+\delta_n)}{(2\delta_n)^{1/2}} \to \infty.$$

It follows that $k \in \bigcup_{m=1}^{\infty} S_m$ and hence

$$0 < \Delta = \mu_{sc}(I) = \mu_{sc}(D) \leqslant \mu_{sc}\left(\bigcup_{m=1}^{\infty} S_m \right) \leqslant \frac{\Delta}{2},$$

a contradiction. □

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